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On Optimal Error Bounds for Derivatives of Interpolating Splines on a Uniform Partition*

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Based on Peano kernel technique, explicit error bounds (optimal for the highest order derivative) are proved for the derivatives of cardinal spline interpolation (interpolating at the knots for odd degree splines and at the midpoints between two knots for even degree splines). The results are based on a new representation of the Peano kernels and on a thorough investigation of their zero distributions. The bounds are given in terms of Euler–Frobenius polynomials and their zeros.

1. INTRODUCTION

Let s be a cardinal spline function of degree n defined over the uniform partition $\mathbb{Z}$ of $\mathbb{R}$ [13]. The cardinal spline s is said to be the cardinal spline interpolation of f at shifted nodes if for a given $v \in [0, 1]$ we have $s(l+v) = f(l+v)$ for all $l \in \mathbb{Z}$. Contributions on the analysis of the cardinal spline interpolation problem have been done by several authors; let us mention the important works of Schoenberg [13–15], de Boor and Schoenberg [2],

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and Subbotin [17, 18]. This paper is a continuation of [7]. Based on new representations (2.4)–(2.5) of Peano kernels, already known results of de Boor and Schoenberg [2] and Schoenberg [15] are generalized and new results are obtained.

The unique solvability of this interpolation problem for functions of polynomial growth is well known (Schoenberg [13, 14], Micchelli [11], ter Morsche [19, 20], Dubeau and Savoie [6]). It happens under the restriction

$$v \neq \tau_n = \begin{cases} 0 \text{ (or 1)} & \text{for } n \text{ even,} \\ \frac{1}{2} & \text{for } n \text{ odd.} \end{cases} \tag{1.1}$$

Let us consider the function spaces

$$L^1_{loc}(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \,\middle|\, \int_a^b |f(x)|\, dx < \infty \text{ for any interval } [a, b] \subset \mathbb{R} \right\}$$

and

$$AC^{n+1}_{loc}(\mathbb{R}) = \left\{ f \in C^n(\mathbb{R}) \,\middle|\, \begin{array}{ll} \text{(i)} & f^{(n+1)} \in L^1_{loc}(\mathbb{R}) \\ \text{(ii)} & \text{for all } [a, b] \subset \mathbb{R},\ f^{(n)}(x)|_a^b = \int_a^b f^{(n+1)}(x)\, dx \end{array} \right\}.$$

A function f is said to be of polynomial growth on $\mathbb{R}$ if there exists an integer $v \geqslant 0$ such that $f(x) = O(|x|^v)$ for $|x| \to +\infty$.

For any $f \in AC^{n+1}_{loc}(\mathbb{R})$, the linear dependence relationships of ter Morsche [19, 20, 5] and the Peano Kernel Theorem [1, 4, 6] lead to

$$p_n(v, E)\, f^{(k)}(l+u) - p_n^k(u, E)\, f(l+v) = \int_0^{n+1} K_n^k(u, v, \theta)\, f^{(n+1)}(l+\theta)\, d\theta$$

for any $l \in \mathbb{Z}$, $u \in [0, 1]$, and $k = 0, ..., n$, where E is the shift operator $Ef(t) = f(t+1)$, $p_n(v, z)$ is the generalized Euler–Frobenius polynomials [3, 19, 20], $p_n^k(u, E) = p_{n-k}(u, E)(E-I)^k$, and

$$K_n^k(u, v, \theta) = p_n(v, E)\, \frac{(u-\theta)_+^{n-k}}{(n-k)!} - p_n^k(u, E)\, \frac{(v-\theta)_+^n}{n!}.$$

Let e be the interpolating error, $e(x) = f(x) - s(x)$, and $e^{(k)}$ be its derivative of order k. Then

$$p_n(v, E)\, e^{(k)}(l+u) = \int_0^{n+1} K_n^k(u, v, \theta)\, f^{(n+1)}(l+\theta)\, d\theta.$$

Moreover, if $f^{(n+1)}$ is of polynomial growth, under (1.1) we obtain

$$e^{(k)}(l+u)=\int_{-\infty}^{\infty} N_n^k(u, v, \theta)\, f^{(n+1)}(l+\theta)\, d\theta, \tag{1.2}$$

where

$$N_n^k(u, v, \theta)=(u-\theta)_+^{n-k}/(n-k)!-p_n(v, E)^{-1}\, p_n^k(u, E)(v-\theta)_+^n/n!. \tag{1.3}$$

Then, using (1.2) for $f^{(n+1)}\in L^\infty(\mathbb{R})$, we obtain

$$|e^{(k)}(l+u)|\leqslant A_n^k(u, v)\, \|f^{(n+1)}\|_\infty \tag{1.4}$$

for any $u\in[0, 1]$ and

$$\|e^{(k)}\|_\infty\leqslant A_n^k(v)\, \|f^{(n+1)}\|_\infty, \tag{1.5}$$

where

$$A_n^k(u, v)=\int_{-\infty}^{\infty} |N_n^k(u, v, \theta)|\, d\theta \tag{1.6}$$

and

$$A_n^k(v)=\sup_{u\in[0, 1]} A_n^k(u, v) \tag{1.7}$$

are the best constants.

For $k=0$ it can be shown [7] that

$$A_n^0(u, v)=\frac{1}{2^{n+1}}\left|\mathscr{E}_{n+2}(u)-\mathscr{E}_{n+1}(u)\,\frac{\mathscr{E}_{n+2}(v)}{\mathscr{E}_{n+1}(v)}\right|$$

and

$$A_n^0(v)=\frac{1}{2^{n+1}}\max_{u\in[0, 1]}\left(|\mathscr{E}_{n+2}(u)|+|\mathscr{E}_{n+1}(u)|\,\frac{|\mathscr{E}_{n+2}(v)|}{|\mathscr{E}_{n+1}(v)|}\right),$$

where $\mathscr{E}_{n+1}$ is the Euler spline of degree n. Moreover,

$$\min_{0\leqslant v\leqslant 1} A_n^0(v)=A_n^0(v_n)=\frac{|\mathscr{E}_{n+2}(v_{n+1})|}{2^{n+1}}, \tag{1.8}$$

where

$$v_n = \begin{cases} \frac{1}{2} & \text{for } n \text{ even,} \\ 0 \text{ (or 1)} & \text{for } n \text{ odd.} \end{cases}$$

Let us point out that de Boor and Schoenberg [2] and Schoenberg [15], using the fundamental cardinal splines, have already obtained (1.2) for $k=0$ and $v=v_n$ and (1.8) for $A_n^0(v_n)$.

For $k>0$, an upper bound for the constants $A_n^k(u, v)$ and $A_n^k(v)$ obtained in [7] is

$$A_n^k(v) \leqslant \frac{(\pi^2/2)^{k-n}}{2\,|\mathscr{E}_{n+1}(v)|}. \tag{1.9}$$

From (1.8) and (1.9) for $v=v_n$, it follows $\|e^{(k)}\|_\infty \leqslant B_n^k \, \|f^{(n+1)}\|_\infty$ for $k=0, ..., n$, where

$$B_n^k = \frac{(\pi^2/2)^k}{2\pi^n}. \tag{1.10}$$

This is a good bound for low values of k, but can be improved for large values of k.

The goal of this paper is twofold. First, we present explicit exact expressions of $A_n^k(u, v_n)$ for particular values of u, and exact values and/or good bounds for $A_n^k(v_n)$. Second, to achieve the first part, we are lead to find new representations for the kernels which are useful to analyse their zero distributions.

In Section 2, new representations (2.4) and (2.5) for the Peano kernels are presented and properties are obtained. These representations are given in terms of the roots of the Euler–Frobenius polynomials in the set $\mathscr{A}_n(v_n) = \{\alpha \in \mathbb{R} \mid p_n(v_n, \alpha) = 0 \text{ and } -1 < \alpha\}$. In Section 3, we obtain the following exact optimal constants for the case $k=n$

$$A_n^n(v_n) = A_n^n(0, v_n) = \frac{1}{2} - \frac{2}{(n+1)} \sum_{\alpha \in \mathscr{A}_n(v_n)} \frac{\alpha}{1+\alpha}. \tag{1.11}$$

To get similar expressions for $A_n^k(u, v_n)$ for specific values of u and to obtain bounds for $A_n^k(v_n)$ for $k=1, ..., n-1$, a thorough investigation of the zero distributions of the Peano kernels is done in Section 4. Finally, the bounds for the cases $k=1, ..., n-1$, are obtained in Section 5. We obtain the relation

$$\begin{aligned} &\max\{A_n^k(v_{n-k}, v_n), A_n^k(v_{n-k+1}, v_n)\} \\ &\qquad \leqslant A_n^k(v_n) \leqslant A_n^k(v_{n-k}, v_n) + A_n^k(v_{n-k+1}, v_n), \end{aligned} \tag{1.12}$$

where $A_n^k(v_{n-k+1}, v_n) = |\mathscr{E}_{n-k+2}(v_{n-k+1})|/2^{n-k+1}$, and the following pointwise optimal error bounds for specific values of u:

(i) for $n-k$ odd,

$$A_n^k(v_{n-k}, v_n) = A_n^k(0, v_n) = \frac{2}{n+1}\left|\sum_{\alpha \in \mathscr{A}_n(v_n)} \frac{p_{n-k}(1, \alpha)}{(1+\alpha)(1-\alpha)^{n-k}}\right|, \tag{1.13}$$

(ii) for $n-k$ even,

$$A_n^k(v_{n-k}, v_n) = A_n^k\left(\frac{1}{2}, v_n\right)$$

$$= \frac{2}{n+1}\left|\sum_{\alpha \in \mathscr{A}_n(v_n)} \frac{\alpha p_{n-k}(1/2, \alpha)}{(1-\alpha)^{n-k}}\left[\frac{1}{1+\alpha} + \frac{p_{n+1}(1-v_n, \alpha) - p_{n+1}(1/2 - v_n, \alpha)}{(1-\alpha)\, p_{n+1}(v_n, \alpha)}\right]\right|. \tag{1.14}$$

Considering the preceding results for $k = 1, ..., n$, we get the bounds

$$\frac{1}{\pi^{n-k+1}} \leqslant A_n^k(v_n) \leqslant 2\,\frac{\sqrt{n+1}}{\pi^{n-k}}. \tag{1.15}$$

This last upper bound is better than B_n^k for large values of k.

EXAMPLE 1.1. As special cases of our results, we obtain the following best constants for quadratic and cubic splines:

(a) $n=2$ and $v = v_2 = \frac{1}{2}$: $A_2^0(\frac{1}{2}) = A_2^0(0, \frac{1}{2}) = \frac{1}{24}$, $A_2^1(\frac{1}{2}) = A_2^1(\frac{1}{2}, \frac{1}{2}) = \frac{1}{8}$, $A_2^2(\frac{1}{2}) = A_2^2(0, \frac{1}{2}) = (1+2\sqrt{2})/6$;

(b) $n=3$ and $v = v_3 = 0$: $A_3^0(0) = A_3^0(\frac{1}{2}, 0) = \frac{5}{384}$, $A_3^1(0) = A_3^1(0, 0) = \frac{1}{24}$, $A_3^2(0) = A_3^2(0, 0) = \sqrt{3}/12$, $A_3^3(0) = A_3^3(0, 0) = (1+\sqrt{3})/4$.

These constants also appear in [12] for periodic quadratic and cubic splines on a uniform partition.

In the next two remarks we present only properties of Euler–Frobenius polynomials used in this paper. For more details look at the indicated references.

Remark 1.2 [3, 8, 19, 20]. We can define the generalized Euler–Frobenius polynomials $p_n(v, z)$ for all $v \in \mathbb{R}$ by the formula

$$p_n(v, z) = \sum_{j=-\infty}^{\infty} Q_n(n+v-j)\, z^j \quad \text{or} \quad \frac{p_n(v, z)}{(1-z)^{n+1}} = \sum_{j=-\infty}^{\infty} \frac{(j+1-v)_+^n}{n!}\, z^j. \tag{1.16}$$

They satisfy the recurrence relation: $p_0(v, z) = 1$, and for $n \geqslant 0$

$$(n+1)\, p_{n+1}(v, z) = [(1-v) + (n+v)\, z]\, p_n(v, z) + z(1-z)\, \partial_z p_n(v, z). \quad (1.17)$$

In the sequel we will use the following consequences of the definition of $p_n(v, z)$: (i) $p_n(v+1, z) = z p_n(v, z)$, (ii) $p_n(v, z) = z^n p_n(1-v, 1/z)$, and $p_n^k(v, z) = (\partial^k/\partial v^k)\, p_n(v, z) = (z-1)^k\, p_{n-k}(v, z)$ for $k = 0, ..., n$.

Remark 1.3 [8, 9]. Let $\mathscr{A}_n(v) = \{\alpha \in \mathbb{R} \mid p_n(v, \alpha) = 0 \text{ and } -1 < \alpha\}$. If $\mathscr{A}_n(v) \neq \varnothing$, set $\alpha_n(v) = \min\{\alpha \mid \alpha \in \mathscr{A}_n(v)\}$ and $\alpha_n = \alpha(v_n)$. The decreasing property of the main roots of the Euler–Frobenius polynomials [8] implies that $\{\alpha_n\}_{n=2}^{\infty}$ is a decreasing sequence of negative numbers lower bounded by -1. Also, if $n \geqslant 2$ and $1 \leqslant k \leqslant n-1$, there exists a unique $u_n^k \in (0, 1)$ such that (i) $p_{n-k}(1-u_n^k, \alpha_n) = 0$ if $n-k$ is odd, and (ii) $p_{n-k}(u_n^k, \alpha_n) = 0$ if $n-k$ is even. Moreover, $u_n^k \in (0, \frac{1}{2})$, and if we set $\sigma_n^k = p_{n-k}(1, \alpha_n)$, then we have $\sigma_n^k = (-1)^{\lfloor (n-k+1)/2 \rfloor}$.

2. PEANO KERNELS: IDENTIFICATION AND PROPERTIES

It is known that $p_n(v, \cdot)$ has no zeros on the unit circle centered at zero for $v \neq \tau_n$. Therefore, $p_n^k(u, z)/p_n(v, z)$ has a Laurent expansion on an annulus containing the unit circle. Let this expansion be

$$p_n^k(u, z)/p_n(v, z) = \sum_{j=-\infty}^{\infty} a_{n,j}^k(u, v)\, z^j, \quad (2.1)$$

where

$$a_{n,j}^k(u, v) = \frac{1}{2\pi i} \int_{C_r} \frac{p_n^k(u, z)}{p_n(v, z)\, z^{j+1}}\, dz, \quad (2.2)$$

C_r is the counterclockwise oriented circle of radius r centered at 0, and $|r-1| < \delta$ for small δ depending on v. Hence, the functions

$$N_n^k(u, v, \theta) = \frac{(u-\theta)_+^{n-k}}{(n-k)!} - \sum_{j=-\infty}^{\infty} a_{n,j}^k(u, v)\, \frac{(v+j-\theta)_+^n}{n!} \quad (2.3)$$

are well defined. In this section we obtain representations and properties of $N_n^k(u, v, \theta)$ based on (2.1), (2.2), (2.3), and properties of Euler–Frobenius polynomials $p_n(v, z)$. In particular, it is shown that if $N_n^k(u, v, \theta)$ is defined by (2.3) it satisfies (1.2). The results we obtain extend those of [2, 15] for $N_n^0(u, v_n, \theta)$.

THEOREM 2.1. *If either $k=n$ and $\theta \neq u$ or $0 \leqslant k < n$, we have*

$$N_n^k(u, v, \theta) = \begin{cases} \dfrac{(u-\theta)_+^{n-k}}{(n-k)!} - \dfrac{1}{2\pi i}\displaystyle\int_{C_{1-\varepsilon}} \dfrac{p_{n-k}(1-u, z)}{p_n(1-v, z)}\, \dfrac{p_n(\theta - v, z)}{(1-z)^{n-k+1}}\, dz, \\ (-1)^{n-k+1}\dfrac{(\theta-u)_+^{n-k}}{(n-k)!} - \dfrac{1}{2\pi i}\displaystyle\int_{C_{1-\varepsilon}} \dfrac{p_{n-k}(u, z)}{p_n(v, z)}\, \dfrac{p_n(v-\theta, z)}{(z-1)^{n-k+1}}\, dz. \end{cases} \tag{2.4}$$

Moreover,

(i) *if either $k=n$ and $u \in (0,1)$ or $0 \leqslant k < n$, then $N_n^k(u, v, j) = 0$ for $j \in \mathbb{Z}$;*

(ii) *if $u \in \{0, 1\}$, then $N_n^n(u, v, j) = 0$ for $j \in \mathbb{Z} - \{u\}$, and $N_n^n(1, v, 1^+) = 0$ and $N_n^n(0, v, 0^-) = 0$.*

Proof. We first observe that using the change of variable $w = 1/z$ and Remark 1.2, (2.2) becomes

$$a_{n,j}^k(u, v) = \frac{1}{2\pi i}\int_{C_{1-\varepsilon}} \frac{p_n^k(1-u, z)\, z^{j-1}}{p_n(1-v, z)}\, dz.$$

Then from (2.3)

$$N_n^k(u, v, \theta) = \frac{(u-\theta)_+^{n-k}}{(n-k)!} - \frac{1}{2\pi i}\int_{C_{1-\varepsilon}} \frac{p_n^k(1-u, z)}{p_n(1-v, z)}\, \frac{\sum_{j\in\mathbb{Z}} (v-\theta+j)_+^n\, z^{j-1}}{n!}\, dz,$$

and from (1.16) we get the first expression for $N_n^k(u, v, \theta)$. In the same way

$$N_n^k(u, v, \theta) = \frac{(u-\theta)_+^{n-k}}{(n-k)!} - \frac{1}{2\pi i}\int_{C_{1+\varepsilon}} \frac{p_{n-k}(u, z)}{p_n(v, z)}\, \frac{p_n(v-\theta, z)}{(z-1)^{n-k+1}}\, dz.$$

Let $z = e^w$. Then

$$\begin{aligned} R &= Res\left[\frac{p_{n-k}(u, z)}{p_n(v, z)}\, \frac{p_n(v-\theta, z)}{(z-1)^{n-k+1}}\right]_{z=1} \\ &= Res\left[e^w\, \frac{p_{n-k}(u, e^w)}{p_n(v, e^w)}\, \frac{p_n(v-\theta, e^w)}{(e^w-1)^{n-k+1}}\right]_{w=0}. \end{aligned}$$

From (1.16), it follows that

$$\frac{p_m(t, e^w)}{(1-e^w)^{m+1}} = \frac{e^{-(1-t)w}}{m!}\, \frac{d^m}{dw^m}\, \frac{e^{(1-t)w}}{1-e^w} = e^{-(1-t)w}\, \frac{(-1)^{m+1}}{w^{m+1}}\, [1 + o(w^{m+1})].$$

Therefore,

$$R = Res\left[\frac{e^{(u-\theta)w}}{w^{n-k+1}}\,[1+o(w^{n-k+1})]\right]_{w=0} = \frac{(u-\theta)^{n-k}}{(n-k)!}.$$

This completes the proof for the second expression in (2.4).

For (i) and (ii), we consider only the case $k=n$ and $u=0$, the other cases are similar or simpler. Using Remark 1.2, the integral in the first line of (2.4) becomes

$$\int_{C_{1-\varepsilon}} \frac{p_0(1,z)}{p_n(1-v,z)}\,\frac{p_n(j-v,z)}{(1-z)^1}\,dz = \int_{C_{1-\varepsilon}} \frac{1}{p_n(1-v,z)}\,\frac{z^{j-1}p_n(1-v,z)}{(1-z)^1}\,dz = 0$$

for $j \geqslant 1$. From (2.4), it follows that (ii) holds for $j \geqslant 1$ and $u=0$. If $j \leqslant 0$, the integral in the second line of (2.4) vanishes; it follows that $N_n^n(0, v, 0^-)=0$ and $N_n^n(0, v, j)=0$ for $j \leqslant -1$. ∎

It is now possible to express the kernels in terms of the exponential splines of Schoenberg [13] (denoted $\mathscr{U}_{n,\alpha}$ in the next theorem).

THEOREM 2.2. *Let*

$$\mathscr{U}_{n,\alpha}(\theta; v) = \frac{p_n(v-\theta, \alpha)}{(\partial/\partial z)\, p_n(v,\alpha)(1-\alpha)},$$

then

$$N_n^k(u, v, \theta)$$
$$= \begin{cases} \dfrac{(u-\theta)_+^{n-k}}{(n-k)!} - \displaystyle\sum_{\alpha \in \mathscr{A}_n(1-v)} \frac{p_{n-k}(1-u,\alpha)}{(1-\alpha)^{n-k}}\,\mathscr{U}_{n,\alpha}(1-\theta; 1-v) & \text{if } \theta > v, \\ (-1)^{n-k+1}\dfrac{(\theta-u)_+^{n-k}}{(n-k)!} + \displaystyle\sum_{\alpha \in \mathscr{A}_n(v)} \frac{p_{n-k}(u,\alpha)}{(\alpha-1)^{n-k}}\,\mathscr{U}_{n,\alpha}(\theta; v) & \text{if } \theta < v. \end{cases} \tag{2.5}$$

Moreover, let $v \neq \tau_n$, and assume $p_n(v,\alpha)=0$. Then

(i) $\mathscr{U}_{n,\alpha}(\theta - j; v) = \alpha^j \mathscr{U}_{n,\alpha}(\theta; v)$,

(ii) $\int_0^1 \mathscr{U}_{n,\alpha}(\theta; v)\, d\theta = 1/(n+1)$,

(iii) $\mathscr{U}_{n,\alpha}(\theta; v) \geqslant 0$ *for* $\theta \in [0, 1]$.

Proof. Using (1.16), we observe that $p_n(\theta - v, z)$ is a polynomial with respect to z if $\theta > v$; this is not the case if $\theta < v$. Hence if $\theta > v$, (2.5) follows from the Residue Theorem and (2.4). Similarly, using (2.4), we can show that (2.5) holds if $\theta < v$. Finally, (i), (ii), and (iii) are direct consequences of Remark 1.2. ∎

As a result we obtain useful identities for $N_n^k(u, v, \theta)$, in particular (vii) below which has already been obtained in [2] for n even and in [15] for n odd.

COROLLARY 2.3. *Let $N_n^k(u, v, \theta)$ be defined for any $u, v, \theta \in \mathbb{R}$ by* (2.4). *Then*

$$\begin{aligned}
&\text{(i)} && N_n^k(u+\nu, v, \theta) = N_n^k(u, v, \theta-\nu) \text{ for } \nu \in \mathbb{Z},\\
&\text{(ii)} && N_n^k(u, v, \theta) = N_n^k(u, v+\nu, \theta) \text{ for } \nu \in \mathbb{Z},\\
&\text{(iii)} && N_n^k(u, v, \theta) = (-1)^{n-k+1} N_n^k(1-u, 1-v, 1-\theta),\\
&\text{(iv)} && N_n^k(u, v_n, \theta) = (-1)^{n-k+1} N_n^k(1-u, v_n, 1-\theta),\\
&\text{(v)} && N_n^k(0, v_n, \theta) = (-1)^{n-k+1} N_n^k(0, v_n, -\theta) \text{ for } k<n,\\
&\text{(vi)} && N_n^0(u, v, \theta) = (-1)^{n+1} N_n^0(\theta-v, 1-v, u-v),\\
&\text{(vii)} && N_n^0(u, v_n, \theta) = (-1)^{n+1} N_n^0(\theta-v_n, v_n, u-v_n).
\end{aligned}$$

We also have

$$\begin{aligned}
&\text{(a)} && \partial_u^l N_n^k(u, v, \theta) = N_n^{k+l}(u, v, \theta) \text{ for } l=0, ..., n-k-1,\\
&\text{(b)} && \partial_u^+ N_n^{n-1}(u, v, \theta) = N_n^n(u^+, v, \theta) \text{ and } \partial_u^- N_n^{n-1}(u, v, \theta) = N_n^n(u^-, v, \theta)
\end{aligned}$$

where $N_n^n(u^+, v, \theta) = N_n^n(u^-, v, \theta)$ for $u \neq \theta$.

Proof. Parts (iii) and (vi) follow directly from (2.4). Parts (i), (ii), and (v) follow from (2.4) and Remark 1.2. Part (iv) follows from (iii) and (ii). Part (vii) follows from (vi) and (ii) if necessary. Finally, (a) and (b) follow from (1.3) and Remark 1.2. ∎

In the next theorem, we prove the exponential decay property of $N_n^k(u, v, \cdot)$. This is a well known property for $v = v_n$ and $k = 0$ [2, 15]. For $k = 1, ..., n$ it could be obtained from the case $k = 0$ and Corollary 2.3(a), (b). We also obtain (1.2) by integration by parts (for $v = v_n$ see also [15, p. 87] for n odd, and [2, p. 45] for n even).

THEOREM 2.4. *Let $v \neq \tau_n$. Then, there exists a constant c such that*

$$|N_n^k(u, v, \theta)| \leqslant c\, |\gamma|^{|\theta|}, \tag{2.6}$$

where $\gamma = \min\{\alpha_n(v), \alpha_n(1-v)\}$ for $\alpha_n(v)$ defined in Remark 1.3. *Also, if s is the cardinal spline interpolation of degree n on $\mathbb{Z}$ of $f \in AC_{loc}^{n+1}(\mathbb{R})$ such that $s(l+v) = f(l+v)$ for $l \in \mathbb{Z}$, then $e = f - s$ verify* (1.2).

Proof. Inequality (2.6) is a direct consequence of Theorem 2.2(i) and (2.5). Let $s(\theta) = N_n^k(u, v, \theta)$ and $w(\theta) = e(l+\theta)$. Using Theorem 2.2(i), (ii) and the fact that $e(l+v) = 0$ for $l \in \mathbb{Z}$, it follows that the function

$s^{(i)}(\theta)\, w^{(n-i)}(\theta)$ is continuous for $i \neq n-k$. In addition, if $i = n-k$, the function has a unique discontinuity at $\theta = u$.

Since s and e are piecewise continuous, and e is of polynomial growth on $\mathbb{R}$, we obtain from integration by parts and (2.6)

$$\int_{-\infty}^{\infty} s(\theta)\, w^{(n+1)}(\theta) = \sum_{i=0}^{n} (-1)^i\, s^{(i)}(\theta)\, w^{(n-i)}(\theta)\big|_{-\infty}^{\infty}.$$

Then (1.2) follows from

$$\begin{aligned}(-1)^{n-k}\, s^{(n-k)}(\theta)\, w^{(k)}(\theta)\big|_{-\infty}^{\infty} &= (-1)^{n-k}\, [s^{(n-k)}(u^+) - s^{(n-k)}(u^-)]\, w^{(k)}(u)\\ &= w^{(k)}(u). \quad \blacksquare\end{aligned}$$

Finally, let us evaluate $G_j(u) = \int_j^{j+1} N_n^k(u, v_n, \theta)\, d\theta$, for $j \in \mathbb{Z}$, in terms of the roots of Euler–Frobenius polynomials. For $j < 0$, from Corollary 2.3(iv) we obtain $G_j(u) = (-1)^{n-k}\, G_{-j}(1-u)$. For $j \geqslant 1$ we use (2.5), $\mathscr{A}_n(1-v_n) - \{0\} = \mathscr{A}_n(v_n)$, and Theorem 2.2(ii) to obtain

$$G_j(u) = -\sum_{\alpha \in \mathscr{A}_n(v_n)} \frac{p_{n-k}(1-u, \alpha)\, \alpha^j}{(n+1)(1-\alpha)^{n-k}}. \tag{2.7}$$

For $j = 0$, we have

$$G_0(u) = \frac{u^{n-k+1}}{(n-k+1)!} - \frac{1}{2\pi i} \int_{C_{1-\varepsilon}} \frac{p_{n-k}(1-u, z)}{p_n(1-v_n, z)} \frac{p_{n+1}(1-v_n, z)}{z(1-z)^{n-k+1}}\, dz.$$

Since $p_l(1-v_n, z) = z p_l(v_n, z)$ for $v_n = 0$ and $p_l(1-v_n, z) = p_l(v_n, z)$ for $v_n = \frac{1}{2}$, then

$$G_0(u) = \frac{u^{n-k+1}}{(n-k+1)!} - \frac{1}{2\pi i} \int_{C_{1-\varepsilon}} \frac{p_{n-k}(1-u, z)}{p_n(v_n, z)} \frac{p_{n+1}(v_n, z)}{z(1-z)^{n-k+1}}\, dz.$$

Using the Residue Theorem, (1.17), and $p_l(t, 0) = (1-t)^l/l!$, we obtain

$$G_0(u) = \frac{u^{n-k+1}}{(n-k+1)!} - \frac{u^{n-k}(1-v_n)}{(n-k)!\,(n+1)} - \sum_{\alpha \in \mathscr{A}_n(v_n)} \frac{p_{n-k}(1-u, \alpha)\, \alpha^j}{(n+1)(1-\alpha)^{n-k}}. \tag{2.8}$$

3. THE CASE $k = n$

In this section we obtain values of the bounds $A_n^n(v_n)$ in terms of the roots of $p_n(v_n, \cdot)$.

Proof of (1.11). From (2.4) we have

$$N_n^n(u, v, \theta) = \begin{cases} f_1(\theta) = \frac{1}{2} - H_n^n(v, \theta) & \text{if} \quad \theta < u, \\ -f_2(\theta) = -\frac{1}{2} - H_n^n(v, \theta) & \text{if} \quad \theta > u, \end{cases}$$

where

$$H_n^n(v, \theta) = \frac{1}{4\pi i} \int_{C_{1-\varepsilon}} \left[\frac{p_n(\theta - v, z)}{p_n(1 - v, z)} - \frac{p_n(v - \theta, z)}{p_n(v, z)} \right] \frac{dz}{(1 - z)} \tag{3.1}$$

which is independent of u. From Theorem 2.1 we have

$$H_n^n(v, j) = \begin{cases} \frac{1}{2} & \text{for} \quad j = 0, -1, -2, ..., \\ -\frac{1}{2} & \text{for} \quad j = 1, 2, ... \end{cases} \tag{3.2}$$

Moreover, from (3.1) we obtain $H_n^n(v, \theta) = -H_n^n(1 - v, 1 - \theta)$ and $H_n^n(0, \theta) = H_n^n(1, \theta)$. It follows that

$$H_n^n(v_n, \tfrac{1}{2} + \theta) = -H_n^n(v_n, \tfrac{1}{2} - \theta). \tag{3.3}$$

Let $\theta_R = l - \delta$ and $\theta_L = 1 - \theta_R$. Using (3.2) and Rolle's Theorem, $\partial_\theta H_n^n(v_n, \theta)$ is a spline of degree $n-1$ having at least $2l-2$ zeros on $(\theta_L, \theta_R) - (0, 1)$. Also, the Budan–Fourier Theorem for splines [15, p. 163, Theorem 4.58] implies $Z_{(\theta_L, \theta_R)}(\partial_\theta H_n^n(v_n, \theta)) \leqslant 2l - 2$. Then, $\partial_\theta H_n^n(v_n, \theta)$ has no zero on (0, 1). Using (3.2), it follows that $H_n^n(v_n, \cdot)$ is decreasing on (0, 1). Since $N_n^n(u, v_n, 0^-) = N_n^n(u, v_n, 1^+) = 0$ and $H_n^n(v_n, \cdot)$ is decreasing, we have from (3.2) that

$$\operatorname{sign} N_n^n(u, v_n, \theta) \quad \begin{cases} > 0 & \text{if} \quad \theta \in (0, u), \\ < 0 & \text{if} \quad \theta \in (u, 1). \end{cases}$$

Using (3.3), we obtain

$$A_n^n(u, v_n) = \int_0^1 |N_n^n(u, v_n, \theta)| \, d\theta + 2 \int_1^\infty |N_n^n(u, v_n, \theta)| \, d\theta.$$

Since $N_n^n(u, v_n, \cdot)$ has no zero on $(j, j+1)$ for $j \in \mathbb{Z} - \{0\}$, we have

$$A_n^n(u, v_n) = \int_0^1 |N_n^n(u, v_n, \theta)| \, d\theta + 2 \left| \sum_{j=1}^{\infty} (-1)^j \int_j^{j+1} N_n^n(u, v_n, \theta) \, d\theta \right|.$$

For the first integral, we have

$$\int_0^1 |N_n^n(u, v_n, \theta)|\, d\theta$$

$$= \int_0^u f_1(\theta)\, d\theta + \int_u^1 f_2(\theta)\, d\theta \leqslant \max\left\{\int_0^1 f_1(\theta)\, d\theta, \int_0^1 f_2(\theta)\, d\theta\right\},$$

where f_1, resp. f_2, is increasing, resp. decreasing (since $H_n^n(v_n, \cdot)$ is decreasing on $(0, 1)$). It follows from (2.8)

$$\int_0^1 |N_n^n(u, v_n, \theta)|\, d\theta \leqslant \int_0^1 |N_n^n(0, v_n, \theta)|\, d\theta$$

$$= -\int_0^1 N_n^n(0, v_n, \theta)\, d\theta = -G_0(0) = \tfrac{1}{2}$$

because the cardinality of the set $\mathscr{A}_n(v_n)$ is $|\mathscr{A}_n(v_n)| = [n-1+2v_n]/2$. Since

$$2\left|\sum_{j=1}^{\infty} (-1)^j \int_j^{j+1} N_n^n(u, v_n, \theta)\, d\theta\right| = 2\sum_{j=1}^{\infty} (-1)^{j+1}\, G_j(u)$$

and $G_j(u) = G_j(0)$ for $j \geqslant 1$, we have

$$A_n^n(u, v_n) \leqslant A_n^n(0, v_n) = -G_0(0) + 2\sum_{j=1}^{\infty} (-1)^{j+1}\, G_j(0). \tag{3.4}$$

Then we obtain (1.11) from (2.7)–(2.8). ∎

Remark 3.1. The sign structure of $N_n^n(u, v_n, \cdot) = \partial_u^n N_n^0(u, v_n, \cdot)$ obtained in the first part of the previous theorem can also be obtained from the result of [2] which states that $N_n^0(\cdot, v_n, \theta)$ changes sign precisely at integers and $\partial_\theta^n N_n^0(u, v_n, \cdot)$ changes sign strongly precisely across each knot, hence in $\mathbb{Z} \cup \{u\}$. The proof can be sketched in two steps. First, using the latter result and Corollary 2.3(i), the sign of $\partial_\theta^n N_n^0(\cdot, v_n, \cdot)$ is deduced. Second, using Corollary 2.3(vii), it follows that $N_n^n(u, v_n, \cdot)$ changes sign strongly at integers and at $\theta = u$.

EXAMPLE 3.2. Exact evaluation of $A_n^n(v_n)$ for $n = 2$ and 3 using (1.11).

(i) For $n = 2$, $p_2 = (\frac{1}{2}, z) = (1 + 6z + z^2)/8$, $\mathscr{A}_2(\frac{1}{2}) = \{-3 + 2\sqrt{2}\}$, and $A_2^2(\frac{1}{2}) = A_2^2(0, \frac{1}{2}) = (1 + 2\sqrt{2})/6$.

(ii) For $n = 3$, $p_3(0, z) = (1 + 4z + z^2)/6$, $\mathscr{A}_3(0) = \{-2 + \sqrt{3}\}$, and $A_3^3(0) = A_3^3(0, 0) = (1 + \sqrt{3})/4$.

To obtain estimates for $A_n^n(v_n)$ we use (1.11), the cardinality of the set $\mathscr{A}_n(v_n)$, and Cauchy–Schwarz inequality to get

$$A_n^n(v_n) = \frac{(1-v_n)}{(n+1)} + \frac{1}{n+1} \sum_{\alpha \in \mathscr{A}_n(v_n)} \frac{1-\alpha}{1+\alpha} \leqslant \frac{(1-v_n)}{(n+1)} + \frac{1}{n+1} |\mathscr{A}_n(v_n)|^{1/2} I_n^{1/2}. \tag{3.5}$$

But from [8]

$$I_n = \sum_{\alpha \in \mathscr{A}_n(v_n)} \left(\frac{1-\alpha}{1+\alpha}\right)^2 \leqslant \frac{8}{\pi^3}(n+1)(n+2).$$

Then we obtain

$$\frac{1}{2} \leqslant A_n^n(v_n) \leqslant \frac{(1-v_n)}{(n+1)} + \frac{2}{\pi^{3/2}} \sqrt{n+1} \leqslant 2\sqrt{n+1}. \tag{3.6}$$

4. THE CASE $1 \leqslant k \leqslant n-1$: ZEROS OF THE KERNEL $N_n^k(u, v_n, \theta)$

This section is concerned with a thorough investigation of the zeros of the kernels $N_n^k(u, v_n, \theta)$ which will be useful to get an upper approximation for $A_n^k(v_n)$. Since we consider only $v = v_n$ in this section, we will use $N_n^k(u, \theta)$ for $N_n^k(u, v_n, \theta)$. Our method to obtain a good approximation is based on the determination of the sign of the kernel $N_n^k(u, \cdot)$, which is directly related to the nature of its zeros.

4.1. *Budan–Fourier Theorem for Hermite–Birkhoff Splines.* The kernels $N_n^k(u, \theta)$ are examples of Hermite–Birkhoff splines (*HB*-splines). In order to obtain information about the zeros of $N_n^k(u, \theta)$, we will use the Budan–Fourier Theorem for *HB*-splines obtained in [10]. This theorem and related notation and definitions are briefly explained in this section.

Let $\Pi : a = x_0 < x_1 < \cdots < x_m = b$ be a partition of the interval $[a, b]$. For m and n given, we denote by F a $(m+1, n+1)$-matrix in which all entries are 0 or 1. F is called an incidence matrix.

We define the space of *HB*-splines by

$$\zeta(F) = \left\{ s : \mathbb{R} \to \mathbb{R} \;\middle|\; \begin{array}{l} s(x) = 0 \text{ for } x \notin [a, b]; \\ s|_{(x_{i-1}, x_i)} \text{ is a polynomial of degree at most } n; \\ s^{(n-j)}(x_i^-) = s^{(n-j)}(x_i^+) \text{ for all } (i, j) \text{ such that } F_{i,j} = 0 \end{array} \right\}.$$

By a block in F we mean a sequence $\{(i, j)\}$, $j = k, ..., k+l-1$, with $F_{i,j} = 1$, and $F_{i,k-1} \neq 1 \neq F_{i,k+l}\}$. The block is called even or odd as l is even or odd. We say the block is supported if there exists i_1, i_2, j_1, j_2 with $i_1 < i < i_2$, $j_1, j_2 < k$ and $F_{i_1, j_1} = F_{i_2, j_2} = 1$. We let $b(F)$ denote the number of supported odd blocks in F.

We now describe how to count the zeros of an *HB*-spline s. For any number c we shall write

$$s(c)^-\ (s(c)^+)=\begin{cases}-1, \\ 0, \\ 1,\end{cases} \quad \text{if} \quad s(x)\begin{cases}<0, \\ =0, \\ >0,\end{cases}$$

on $(c-\delta, c)$ (or $(c, c+\delta)$) for some $\delta>0$.

Now suppose we have numbers $\alpha \leqslant \beta$ such that $f(x)=0$ on (α, β) and $s(\alpha)^- \neq 0 \neq s(\beta)^+$. Define $l \geqslant 0$, $r \geqslant 0$ by $s(\alpha^-)=\cdots=s^{(l-1)}(\alpha^-)=0$, $s^{(l)}(\alpha)^- \neq 0$, $s(\beta^+)=\cdots=s^{(r-1)}(\beta^+)=0$, $s^{(r)}(\beta)^+ \neq 0$, and set $s=\min(l, r)$. Then we say that $[\alpha, \beta]$ is a zero of s of multiplicity M, where

$$M=\begin{cases}m, & \text{if} \quad \alpha=\beta \text{ and } s(\alpha)^-\ s(\beta)^+=(-1)^m, \\ m+1, & \text{otherwise.}\end{cases}$$

Moreover, if $\alpha<\beta$ we say that $[\alpha, \beta]$ is an interval-zero of s of multiplicity M. Finally, an interval I such that $I=[a, \beta)$ or $(\alpha, b]$, where $s(x)=0$ on I, is said an interval-zero of multiplicity 0.

Let $Z_{(a, b)}(s)$ be the number of zeros of s on (a, b) counting multiplicity. For a real vector $w=(w_0, ..., w_n)$, let S^-w, resp. S^+w be the minimal, resp. maximal number of sign changes in the sequence w achievable by appropriate assignment of signs to the zero entries of w. If $S^-w=S^+w$, we denote their common value by Sw.

THEOREM 4.1 [10, Theorem 2.1, p. 453]. *If $s \in \zeta(F)$ has exact degree n (that is, $s^{(n)}$ is not the zero function and $s^{(n+1)}(x)=0$ for all x), then*

$$Z_{(a, b)}(s) \leqslant S^-(s(a^+), ..., s^{(n)}(a^+))-S^+(s(b^-), ..., s^{(n)}(b^-)) + \sum_{i=1}^{m-1} \sum_{j=0}^{n} F_{i, j}+b(F).$$

4.2. *Peano Kernels $N_n^k(u, \cdot)$ as HB-Splines.* From the representation (2.3) it follows that $N_n^k(u, \cdot)$ is a *HB*-spline of degree n with knots $(v_n+\mathbb{Z}) \cup \{u\}$. Considering the zeros of $N_n^k(u, \cdot)$ as defined for *HB*-splines, we observe that the zeros of $N_n^k(u, \cdot)$ are either usual zeros of a function $(l=r=m)$ or interval-zeros.

Let $b=\theta_R=l-\delta$ and $a=\theta_L=1-\theta_R$ for $l \geqslant 1$ and $\delta>0$. The incidence matrix F has the following form for the different cases n (v_n) and u:

(i) n odd $(v_n=0)$. For $u \neq v_n$, F is a $(2l+1, n+1)$-matrix such that $\sum_{i, j} F_{i, j}=2l-1$ and $b(F)=1$. For $u=v_n$, F is a $(2l, n+1)$-matrix such that $\sum_{i, j} F_{i, j}=2l-1$, and $b(F)=0$ for $k=1$, $b(F)=1$ for $k>1$.

(ii) n even ($v_n = \frac{1}{2}$). For $u \neq v_n$, F is a $(2l+2, n+1)$-matrix such that $\sum_{i,j} F_{i,j} = 2l$ and $b(F) = 1$. For $u = v_n$, F is a $(2l+1, n+1)$-matrix such that $\sum_{i,j} F_{i,j} = 2l$, and $b(F) = 0$ for $k = 1$, $b(F) = 1$ for $k > 1$.

In summary, we have

$$\sum_{i,j} F_{i,j} + b(F) = 2l + 1 - \delta_1^k \delta_{v_n}^u - n(\text{mod } 2), \tag{4.1}$$

where $\delta_\alpha^\beta = 1$ if $\alpha = \beta$ or 0 if $\alpha \neq \beta$.

4.3. *The Number of Zeros of* $N_n^k(u, \cdot)$. We already know from Section 2 that $N_n^k(u, l) = 0$ for $l \in \mathbb{Z}$ but we did not show that those zeros are isolated or not. The main result of this section is Theorem 4.5 in which it is shown that almost all kernels $N_n^k(u, \cdot)$ have only isolated zeros (no interval-zero) and at most one more zero which depends on u.

Using Theorem 4.1 where the count of zeros of $N_n^k(u, \cdot)$ is done on an interval $[a, b] = [\theta_L, \theta_R]$, we obtain Theorem 4.5 in three steps. First, in Theorem 4.2 we analyse the sign structure of the kernel $N_n^k(u, \cdot)$. Second, in Theorem 4.3 we obtain the cases for which an interval-zero exists. Third, an upper bound and a lower bound are given in Theorem 4.4 for the number of zeros.

THEOREM 4.2. *Let*

(i) $r_i(v, \zeta)$ *be the number of roots of* $p_i(v, z)$ *which are greater than* ζ *where* $-1 < \zeta < 0$,

(ii) $\theta_R = l - \delta$, $\theta_L = 1 - \theta_R$ *where* l *is a sufficiently large integer and* $\delta > 0$, *and*

(iii) u_n^k *as defined in Remark* 1.3.

The kernel $N_n^k(u, \cdot)$ *has the following sign properties.*

(1) *Let* ω_L *be the smallest value of* $\mathscr{A}_n(v_n)$ *such that* $p_{n-k}(u, \omega_L) \neq 0$, *and* $\delta > 0$ *be sufficiently small. If* ω_L *exists, then*

$$\text{sign}[N_n^k(u, \theta_L)] = (-1)^{n-k+1+l} \, \text{sign}[p_{n-k}(u, \omega_L)], \tag{4.2a}$$

there exists a constant $C_L = \pm 1$ *independent of* j *and* l *such that*

$$\text{sign}[\partial_\theta^j N_n^k(u, \theta_L)] = C_L(-1)^l \, \text{sign}[p_{n-j}(1 - \delta - v_n, \omega_L)], \tag{4.2b}$$

and

$$S_L^- = S^-(N_n^k(u, \theta_L), \partial_\theta N_n^k(u, \theta_L), ..., \partial_\theta^n N_n^k(u, \theta_L)) = r_n(1 - v_n, \omega_L). \tag{4.3}$$

(2) *Let* ω_R *be the smallest value of* $\mathscr{A}_n(1-v_n)$ *such that* $p_{n-k}(1-u, \omega_R) \neq 0$, *and* $\delta > 0$ *be sufficiently small. If* ω_R *exists, then*

$$\operatorname{sign}[N_n^k(u, \theta_R)] = (-1)^l \operatorname{sign}[p_{n-k}(1-u, \omega_R)], \tag{4.4a}$$

there exists a constant $C_R = \pm 1$ *independent of* j *and* l *such that*

$$\operatorname{sign}[\partial_\theta^j N_n^k(u, \theta_R)] = C_R(-1)^{j+l} \operatorname{sign}[p_{n-j}(1-\delta-v_n, \omega_R)], \tag{4.4b}$$

and

$$S_R^+ = S^+(N_n^k(u, \theta_R), \partial_\theta N_n^k(u, \theta_R), ..., \partial_\theta^n N_n^k(u, \theta_R)) = n - r_n(1-v_n, \omega_R). \tag{4.5}$$

Moreover,

(i) *if* $u \notin \{u_n^k, 1-u_n^k\}$ *then* $\alpha_n = \omega_R = \omega_L$;

(ii) *if* $\omega_R \neq \alpha_n$, *resp.* $\omega_L \neq \alpha_n$, *then* $\omega_L = \alpha_n$, *resp.* $\omega_R = \alpha_n$.

Proof. From (2.5),

$$\partial_\theta^j N_n^k(u, \theta_R) = (-1)^{j+1} \sum_{\alpha \in \mathscr{A}_n(1-v_n)} \frac{p_{n-k}(1-u, \alpha)}{(1-\alpha)^{n+1-k-j}} \frac{p_{n-j}(1-\delta-v_n, \alpha)}{\partial_z p_n(1-v_n, \alpha)} \alpha^{l-1}.$$

For large enough l, we obtain

$$\begin{aligned}
&\operatorname{sign}[\partial_\theta^j N_n^k(u, \theta_R)] \\
&\quad = \operatorname{sign}\left[(-1)^{j+1} \frac{p_{n-k}(1-u, \omega_R)}{(1-\omega_R)^{n+1-k-j}} \frac{p_{n-j}(1-\delta-v_n, \omega_R)}{\partial_z p_n(1-v_n, \omega_R)} \omega_R^{l-1}\right] \\
&\quad = C_R \operatorname{sign}[(-1)^{j+l} p_{n-j}(1-\delta-v_n, \omega_R)],
\end{aligned}$$

where $C_R = \pm 1$ but is independent of j and l. Hence, (4.4b) follows. Using a similar argument, (4.4a) follows from (2.5) and Theorem 2.2. It follows from Corollary 2.3(iv) that $\partial_\theta^j N_n^k(u, \theta_L) = (-1)^{n-k+1+j} \partial_\theta^j N_n^k(1-u, \theta_R)$, and (4.2) holds. As a direct consequence of the interlacing properties of the roots of $p_n(v, z)$ we have

$$S^-(p_0(v, \zeta), ..., p_k(v, \zeta)) = r_k(v, \zeta) \tag{4.6}$$

from which we obtain

$$\begin{aligned}
S_R^+ &= n - S^-(N_n^k(u, \theta_R), ..., (-1)^n \partial_\theta^n N_n^k(u, \theta_R)) \\
&= n - S^-(p_n(1-\delta-v_n, \omega_R), ..., p_0(1-\delta-v_n, \omega_R)) \\
&= n - r_n(1-\delta-v_n, \omega_R).
\end{aligned}$$

This implies (4.5). The proof of (4.3) is similar. We obtain (i) and (ii) directly from Remark 1.3. ∎

The occurrence of an interval-zero is based on the following two observations [9]. First, assume $I=[\alpha, \beta]$ is an interval-zero of $N_n^k(u, \cdot)$: if $\beta > u$ then (i) ω_R does not exist, $\omega_L = \alpha_n$, and $u = 2v_{n-k} + (-1)^{n-k+1} u_n^k$, (ii) $[\max\{1-v_n, u\}, \infty) \subset I$, (iii) $\alpha \in \{u, \max\{1-v_n, u\}\}$; if $\alpha < u$ then (iv) ω_L does not exist, $\omega_R = \alpha_n$, and $u = 2v_{n-k+1} + (-1)^{n-k} u_n^k$, (v) $(-\infty, \min\{v_n, u\}] \subset I$, (vi) $\beta \in \{u, \min\{v_n, u\}\}$. This fact can be proved by considering $N_n^k(u, \cdot)$ as a *HB*-spline on the interval $[a, b] = [\theta_L, \theta_R]$ with knots $U = [(v_n + \mathbb{Z}) \cup \{u, a, b\}] \cap [a, b]$. Second, if $N_n^k(u, \cdot)$ has no interval-zero, then there exists $\omega_L \in \mathscr{A}_n(v_n)$ such that $p_{n-k}(u, \omega_L) \neq 0$, and there exists $\omega_R \in \mathscr{A}_n(1-v_n)$ such that $p_{n-k}(1-u, \omega_R) \neq 0$.

THEOREM 4.3. *$N_n^k(u, \cdot)$ has no interval-zero I on $\mathbb{R}$ if and only if $n=2$ or 3 and $u \in \{u_n^k, 1-u_n^k\}$. Moreover, when an interval-zero exists*

$$I = \begin{cases} [1-v_n, \infty) & \text{if} \quad u = 2v_{n-k} + (-1)^{n-k+1} u_n^k, \\ (-\infty, v_n] & \text{if} \quad u = 2v_{n-k+1} + (-1)^{n-k} u_n^k, \end{cases}$$

and all isolated zeros of $N_n^k(u, \cdot)$ are simple and in $\mathbb{Z}$.

Proof. If $N_n^k(u, \cdot)$ has an interval-zero, it follows from the first observation that $u \in \{u_n^k, 1-u_n^k\}$. Assume $u = u_n^k$ and ω_R does not exist (the other cases are similar). Then, $\omega_L = \alpha_n$, $N_n^k(u_n^k, \cdot)$ is of exact degree n on $(-\infty, u_n^k]$, and $N_n^k(u_n^k, \theta) = 0$ for $\theta \in [t, \infty)$ where $0 < t = \max\{1-v_n, u\} = 1-v_n$. From Theorem 4.2, we have

$$S^-(N_n^k(u_n^k, \theta_L), ..., \partial_\theta^n N_n^k(u_n^k, \theta_L)) = r_n(1-v_n, \alpha_n)$$

and

$$S^+(N_n^k(u_n^k, t), ..., \partial_\theta^n N_n^k(u_n^k, t)) = n$$

since at most $\partial_\theta^n N_n^k(u_n^k, t)$ is non-zero. It follows

$$S_{\theta_L}^- - S_t^+ = r_n(1-v_n, \alpha_n) - n = \begin{cases} -\dfrac{n+2}{2} & \text{if} \quad n \text{ is even} \\ -\dfrac{n+1}{2} & \text{if} \quad n \text{ is odd.} \end{cases}$$

Considering the *HB*-spline $N_n^k(u_n^k, \cdot)$ on the interval (θ_L, t), it follows that

$$\sum_i \sum_j F_{i,j} + b(F) \leqslant l+1.$$

Moreover, from (iii) of the first observation, $N_n^k(u_n^k, j)=0$ for $j=0, -1, ..., 2-l$ and these zeros are isolated. Hence,

$$l-1 \leqslant Z_{(\theta_L, t)}(N_n^k(u_n^k, \cdot)) \leqslant \begin{cases} l-\dfrac{n}{2} & \text{if } n \text{ is even,} \\ l-\dfrac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

This implies that $n \leqslant 2$ if n is even, and $n \leqslant 3$ if n is odd. Moreover, in these two cases it follows $Z_{(\theta_L, t)}(N_n^k(u_n^k, \cdot))=l-1$ which implies that all isolated zeros are simple and element of $\mathbb{Z}$.

Conversely, if $n=2$ or 3 and $u \in \{u_n^k, 1-u_n^k\}$, we present explicit expressions for $N_n^k(u, \cdot)$. For $n=2$ and $k=1$ we have

$$N_2^1(u_2^1, \theta)=\begin{cases}(\theta-u_2^1)_+ - 2\sqrt{2}\, u_2^1(\theta+\sqrt{2}\,\theta^2) & \text{for } 0 \leqslant \theta \leqslant \frac{1}{2}, \\ 0 & \text{for } \theta > \frac{1}{2},\end{cases}$$

where $u_2^1=(2-\sqrt{2})/4$, hence $I=[\frac{1}{2}, \infty)$. For $n=3$ and $k=2$ we have

$$N_3^2(u_3^2, \theta)=\begin{cases}(\theta-u_3^2)_+ - u_3^2(1-\theta)^3 & \text{for } 0 \leqslant \theta \leqslant 1, \\ 0 & \text{for } \theta > 1,\end{cases}$$

where $u_3^2=(3-\sqrt{3})/6$, hence $I=[1, \infty)$. For $n=3$ and $k=1$ we have

$$N_3^1(1-u_3^1, \theta)=\begin{cases}((1-u_3^1)-\theta)_+^2/2-(1-u_3^1)^2\,(1-\theta)^3/2 & \text{for } 0 \leqslant \theta \leqslant 1, \\ 0 & \text{for } \theta > 1,\end{cases}$$

where $u_3^1=1/2+\sqrt{3}\,(1-\sqrt{2})/6$, hence $I=[1, \infty)$. ∎

THEOREM 4.4. *If $N_n^k(u, \cdot)$ has no interval-zero, then $Z_{(\theta_L, \theta_R)}(N_n^k(u, \cdot)) \geqslant 2l-2$. Moreover,*

(i) *for $n \geqslant 2$, if $u \notin \{u_n^k, 1-u_n^k\}$ then $Z_{(\theta_L, \theta_R)}(N_n^k(u, \cdot)) \leqslant 2l-1$.*

(ii) *for $n \geqslant 4$, if $u \in \{u_n^k, 1-u_n^k\}$ then $Z_{(\theta_L, \theta_R)}(N_n^k(u, \cdot))=2l-2$. Also, either ω_R or ω_L is equal to α_n, the other one is the second lowest value of $\mathscr{A}_n(v_n)$.*

As a consequence $N_n^k(u, \cdot)$ has an even, resp. odd, number of sign change on (θ_L, θ_R) if $Z_{(\theta_L, \theta_R)}(N_n^k(u, \cdot))$ is even, resp. odd.

Proof. If $N_n^k(u, \cdot)$ has no interval-zero, the zeros of $N_n^k(u, \cdot)$ are isolated. Since $N_n^k(u, j)=0$ for $j=-l+2, ..., l-1$, then $Z_{(\theta_L, \theta_R)}(N_n^k(u, \cdot)) \geqslant 2l-2$.

To obtain (i) we use Theorem 4.2 for $u \notin \{u_n^k, 1-u_n^k\}$. It follows that $S_L^- = r_n(1-v_n, \alpha_n) = n - S_R^+$. Therefore,

$$S_L^- - S_R^+ = -n + 2r_n(1-v_n, \alpha_n) = \begin{cases} -2 & \text{if} \quad n \text{ is even;} \\ -1 & \text{if} \quad n \text{ is odd.} \end{cases} \tag{4.7}$$

The result follows from (4.1), Theorem 4.1, and (4.7). To obtain (ii), let us suppose that $n-k$ is even (the case $n-k$ odd is similar). We have $p_{n-k}(u_n^k, \alpha_n) = 0$ and $p_{n-k}(1-u_n^k, \alpha_n) \neq 0$. Hence, $\omega_R = \alpha_n$ and since $N_n^k(u_n^k, \cdot)$ has no interval-zero it follows that ω_L exists by the second observation. From Theorem 4.2

$$S_L^- - S_R^+ = -n + r_n(1-v_n, \alpha_n) + r_n(1-v_n, \omega_L) = \begin{cases} -2-j & \text{if} \quad n \text{ is even} \\ -1-j & \text{if} \quad n \text{ is odd,} \end{cases}$$

where j is the number of elements in $\mathscr{A}_n(v_n)$ smaller than ω_L. Therefore, we obtain

$$Z_{(\theta_L, \theta_R)}(N_n^k(u_n^k, \cdot)) \leqslant 2l - 1 - j.$$

It follows that $j = 1$. Finally, if $N_n^k(u, \cdot)$ has only isolated zeros, the counting procedure of zeros used here preserves the parity between the number of sign changes and the number of zeros on an interval. *If* $N_n^k(u, \cdot)$ has an interval-zero, it is a end interval-zero from the first observation. Since these intervals have multiplicity 0 from the definition, the desired result holds also in that case. ∎

From Corollary 2.3, the kernels $N_n^k(u, \cdot)$ have the following special properties:

(i) $N_n^k(0, \cdot)$ is an even function for $n-k$ odd and an odd function for $n-k$ even. It follows that $\theta = 0$ is a double zero of $N_n^k(0, \cdot)$ for $n-k$ odd, and $Z_{(\theta_L, \theta_R)}(N_n^k(0, \cdot)) = 2l - 2$ for $n-k$ even.

(ii) $N_n^k(\frac{1}{2}, \frac{1}{2} + \cdot)$ is an even function for $n-k$ odd and an odd function for $n-k$ even. It follows that $\theta = \frac{1}{2}$ is a simple zero of $N_n^k(\frac{1}{2}, \cdot)$ for $n-k$ even, and $Z_{(\theta_L, \theta_R)}(N_n^k(\frac{1}{2}, \cdot)) = 2l - 2$ for $n-k$ odd.

Let

$$\mathscr{I}_n^k = \left\{ u \in [0, 1] \,\middle|\, \begin{array}{l} N_n^k(u, \cdot) \text{ has no interval-zero on } \mathbb{R}, \text{ and} \\ \text{there exists } [\theta_L, \theta_R] \text{ such that } Z_{(\theta_L, \theta_R)}(N_n^k(u, \cdot)) = 2l - 1 \end{array} \right\},$$

$$\mathscr{J}_n^k = \left\{ u \in [0, 1] \,\middle|\, \begin{array}{l} N_n^k(u, \cdot) \text{ has no interval-zero on } \mathbb{R}, \text{ and} \\ \text{for all } [\theta_L, \theta_R] \text{ we have } Z_{(\theta_L, \theta_R)}(N_n^k(u, \cdot)) = 2l - 2 \end{array} \right\}.$$

We have

$$\mathscr{I}_n^k \cap \mathscr{J}_n^k = \varnothing, \tag{4.8a}$$

$$\text{for} \quad n = 2 \text{ or } 3, \qquad [0, 1] - \mathscr{I}_n^k \cup \mathscr{J}_n^k = \{u_n^k, 1 - u_n^k\}, \tag{4.8b}$$

$$\text{for} \quad n \geqslant 4, \qquad \mathscr{I}_n^k \cup \mathscr{J}_n^k = [0, 1]. \tag{4.8c}$$

The preceding results and Corollary 2.3 lead to the following theorem.

THEOREM 4.5. *The zeros of* $N_n^k(u, \cdot)$ *are related to the intervals* $\mathscr{I}_n^k$ *and* $\mathscr{J}_n^k$ *in the following way.*

(i) *If* $u \in \mathscr{J}_n^k$, *then* $\mathbb{Z}$ *is the set of zeros of* $N_n^k(u, \cdot)$, *and these zeros are simple and isolated.*

(ii) *If* $u \in \mathscr{I}_n^k$, *then the set of zeros of* $N_n^k(u, \cdot)$ *is* $\mathbb{Z} \cup \{\psi_n^k(u)\}$. *If* $\psi_n^k(u) \notin \mathbb{Z}$, *then all the zeros are simple and isolated. If* $\psi_n^k(u) \in \mathbb{Z}$, $\psi_n^k(u)$ *is a double zero and all other zeros are simple. Also,*

$$\psi_n^k(u) + \psi_n^k(1 - u) = 1. \tag{4.9}$$

(iii) *If* $n = 2$ *or* 3, *and* $u \in \{u_n^k, 1 - u_n^k\}$ *then* $N_n^k(u, \theta) = 0$ *on* $\mathbb{Z} \cup I$ *where* I *is of the form* $(-\infty, v_n]$ *or* $[1 - v_n, \infty)$ (*see Theorem* 4.3) *and* $\mathbb{Z} - I$ *is the set of isolated simple zeros.*

4.4. *The Sign of the Kernel* $N_n^k(u, \theta)$. Let $u \notin \{u_n^k, 1 - u_n^k\}$, and consider l sufficiently large. It follows from Theorem 4.2 that

$$\operatorname{sign} N_n^k(u, \theta_R) = (-1)^l \operatorname{sign} p_{n-k}(1 - u, \alpha_n),$$

$$\operatorname{sign} N_n^k(u, \theta_L) = (-1)^{n-k+1+l} \operatorname{sign} p_{n-k}(u, \alpha_n).$$

Hence

$$\operatorname{sign} N_n^k(u, \theta) = \begin{cases} (-1)^l \operatorname{sign} p_{n-k}(1 - u, \alpha_n) & \text{for} \quad \theta \in (l - 1, l), \\ (-1)^{n-k+1+l} \operatorname{sign} p_{n-k}(u, \alpha_n) & \text{for} \quad \theta \in (1 - l, 2 - l) \end{cases} \tag{4.10}$$

for sufficiently large l. It follows

$$\operatorname{sign}[N_n^k(u, \theta_L)\, N_n^k(u, \theta_R)] = (-1)^{n-k+1} \operatorname{sign}[p_{n-k}(1 - u, \alpha_n)\, p_{n-k}(u, \alpha_n)].$$

But

$$\operatorname{sign}[p_{n-k}(1 - u, \alpha_n)\, p_{n-k}(u, \alpha_n)] = \begin{cases} 1 & \text{if} \quad u \in (u_n^k, 1 - u_n^k), \\ -1 & \text{if} \quad u \in [0, u_n^k) \cup (1 - u_n^k, 1]. \end{cases}$$

Hence,

$$\operatorname{sign}[N_n^k(u,\theta_L)\,N_n^k(u,\theta_R)]=\begin{cases}(-1)^{n-k+1} & \text{if } u\in(u_n^k,1-u_n^k),\\ (-1)^{n-k} & \text{if } u\in[0,u_n^k)\cup(1-u_n^k,1].\end{cases}$$

Using the last equation, and the last statement of Theorem 4.4, it follows that

$$\mathscr{I}_n^k=\begin{cases}(u_n^k,1-u_n^k) & \text{if } n-k \text{ is even},\\ [0,u_n^k)\cup(1-u_n^k,1] & \text{if } n-k \text{ is odd},\end{cases}$$

and $\mathscr{J}_n^k$ is defined to satisfy (4.8).

Using (4.10), Remark 1.3, and Theorem 4.4, we obtain the sign of $N_n^k(u,\theta)$ as indicated in Figs. 4.1 and 4.2.

THEOREM 4.6. *Let $\tau(\theta)=(-1)^l$ if $\theta\in(l,l+1)$. If $u\in\mathscr{I}_n^k$ and $\theta\notin\mathbb{Z}$,*

$$\operatorname{sign} N_n^k(u,\theta)=\begin{cases}-\tau(\theta)\operatorname{sign} p_{n-k}(1-u,\alpha_n) & \text{for } \theta>\psi_n^k(u),\\ (-1)^{n-k}\tau(\theta)\operatorname{sign} p_{n-k}(u,\alpha_n) & \text{for } \theta<\psi_n^k(u)\end{cases}\tag{4.11}$$

and if $u\in\mathscr{J}_n^k$,

$$\operatorname{sign} N_n^k(u,\theta)=-\tau(\theta)\operatorname{sign} p_{n-k}(1-u,\alpha_n)=(-1)^{n-k}\tau(\theta)\operatorname{sign} p_{n-k}(u,\alpha_n).$$

Proof. Since the sign changes of $\tau(\theta)$ and $N_n^k(u,\theta)$ occur respectively at each $\theta\in\mathbb{Z}$ and $\theta\in\mathbb{Z}\cup\{\psi_n^k(u)\}$ or $\theta\in\mathbb{Z}$, we obtain the result from (4.10). ∎

4.5. *Properties of ψ_n^k.* In this section the monotonicity and differentiability properties of ψ_n^k are established in Theorem 4.14. To obtain these properties we analyse the inverse function μ_n^k of ψ_n^k.

THEOREM 4.7. *Let $n-k$ be even. There exists a unique increasing continuous function $\mu_n^k\colon[\frac{1}{2},\infty)\to[\frac{1}{2},1-u_n^k]$ such that $\mu_n^k(\psi_n^k(u))=u$ for $u\in[\frac{1}{2},1-u_n^k)$. Moreover, $\mu_n^k(\frac{1}{2}^+)=\frac{1}{2}$, and*

(i) *if $n=3$, μ_3^1 is strictly increasing on $[\frac{1}{2},1]$ and $\mu_3^1(\theta)=1-\mu_3^1$ for $\theta\geqslant 1$ (see Fig. 4.3);*

(ii) *if $n\geqslant 4$, μ_n^k is strictly increasing on $[\frac{1}{2},\infty)$ and $\lim_{\theta\to\infty}\mu_n^k(\theta)=1-u_n^k$ (see Fig. 4.4).*

We can extend μ_n^k on $(-\infty,\frac{1}{2}]$ such that $\mu_n^k(\frac{1}{2}-\theta)=1-\mu_n^k(\frac{1}{2}+\theta)$.

Proof. Let $\theta\in(1,\infty)$ and $\theta\notin\mathbb{Z}$. Then $N_n^k(\cdot,\theta)$ is a polynomial of degree $n-k$. Considered as a *HB*-spline, its incidence matrix in $[0,1]$ has two

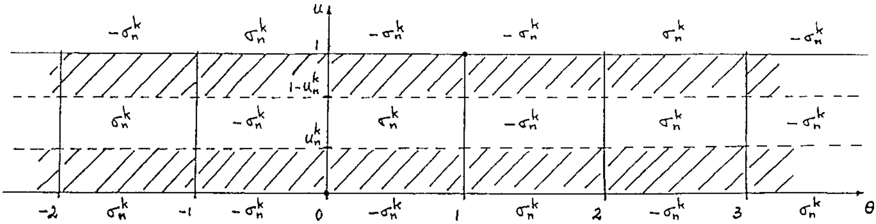

FIG. 4.1. Sign of $N_n^k(u, \theta)$ for $n-k$ odd and $\sigma_n^k = (-1)^{(n-k+1)/2}$.

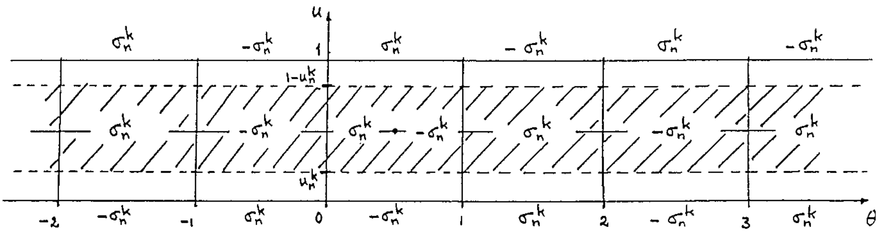

FIG. 4.2. Sign of $N_n^k(u, \theta)$ for $n-k$ even and $\sigma_n^k = (-1)^{(n-k)/2}$.

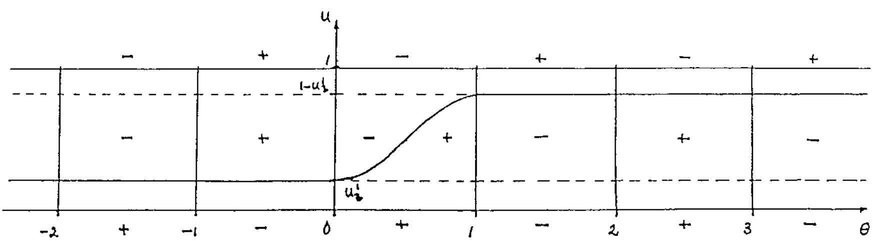

FIG. 4.3. Sketch of μ_3^1.

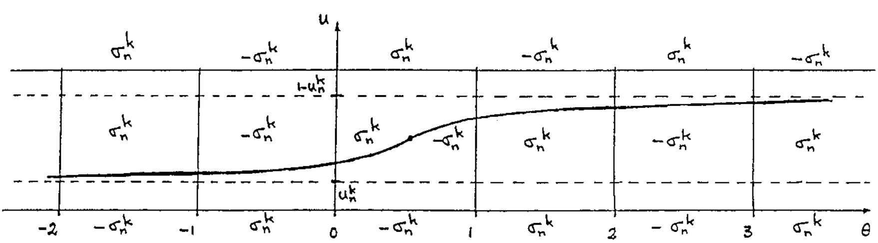

FIG. 4.4. Sketch of μ_n^k for $n \geqslant 4$ and $n-k$ even.

columns. Then $\sum_{i,j} F_{i,j}=0$ and $b(F)=0$. Also from Corollary 2.3(a),(b), Theorem 4.6, and (4.6), we obtain

$$S_0^- = S(p_{n-k}(1, \alpha_n), p_{n-k-1}(1, \alpha_n), ..., p_1(1, \alpha_n), p_0(1, \alpha_n)),$$

$$S_1^+ = S(p_{n-k}(0, \alpha_n), p_{n-k-1}(0, \alpha_n), ..., p_1(0, \alpha_n), p_0(0, \alpha_n)),$$

and $S_0^- - S_1^+ = 1$. Then from Theorem 4.1, $0 \leqslant Z_{(0,1)}(N_n^k(\cdot, \theta)) \leqslant 1$. Moreover, since $\text{sign}[N_n^k(\frac{1}{2}, \theta)\, N_n^k(1, \theta)] = -1$, then $Z_{(0,1)}(N_n^k(\cdot, \theta)) = 1$. Hence $\mu_3^1(\theta) = 1 - u_3^1$, and for $n \geqslant 4$, $\mu_n^k(\theta) \in (\frac{1}{2}, 1 - u_n^k)$.

For $\theta \in (\frac{1}{2}, 1)$, $N_n^k(\cdot, \theta)$ is a *HB*-spline of degree $n-k$ with a unique knot at $u=\theta$ on $[0, 1]$. Its incidence matrix is such that $\sum_{i,j} F_{i,j}=1$ and $b(F)=0$. Also

$$\begin{aligned} S_0^- &= S(p_{n-k}(1, \alpha_n), p_{n-k-1}(1, \alpha_n), ..., p_1(1, \alpha_n), p_0(1, \alpha_n)) \\ S_1^+ &= S((-1)^{n-k}\, p_{n-k}(1, \alpha_n), (-1)^{n-k-1}\, p_{n-k-1}(1, \alpha_n), ..., \\ &\qquad -p_1(1, \alpha_n), p_0(1, \alpha_n)) \\ &= n-k-S(p_{n-k}(1, \alpha_n), p_{n-k-1}(1, \alpha_n), ..., p_1(1, \alpha_n), p_0(1, \alpha_n)). \end{aligned}$$

Hence, we have $S_0^- - S_1^+ = 0$ for $n-k$ even. Then $0 \leqslant Z_{(0,1)}(N_n^k(\cdot, \theta)) \leqslant 1$. Moreover, since

$$\text{sign}[N_n^k(\tfrac{1}{2}, \theta)\, N_n^k(1, \theta)] = -1,$$

then $Z_{(0,1)}(N_n^k(\cdot, \theta)) = 1$ and $\mu_n^k(\theta) \in (\frac{1}{2}, 1 - u_n^k)$. Lemmas 4.9 and 4.10 complete the proof. ∎

THEOREM 4.8. *Let $n-k$ be odd. There exists a unique increasing continuous function $\mu_n^k\colon [0, \infty) \to [0, u_n^k]$ such that $\mu_n^k(\psi_n^k(u)) = u$ for $u \in [0, u_n^k)$. Moreover, $\mu_n^k(0^+) = 0$, and*

(i) *if $n=2$, μ_2^1 is strictly increasing on $[0, \frac{1}{2}]$ and $\mu_2^1(\theta) = u_2^1$ for $\theta > \frac{1}{2}$* (*see Fig.* 4.5);

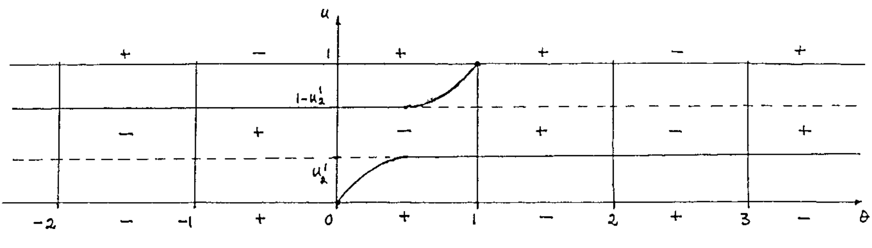

FIG. 4.5. Sketch of μ_2^1.

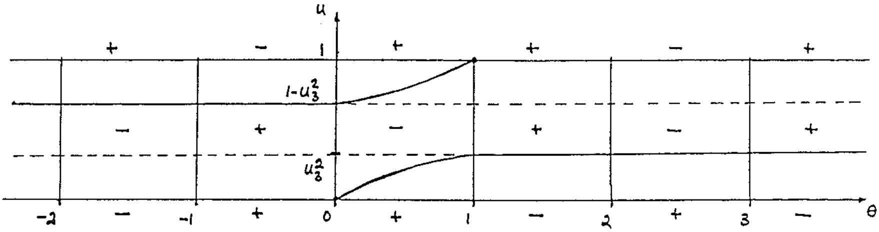

FIG. 4.6. Sketch of μ_3^2.

(ii) *if* $n=3$, μ_3^2 *is strictly increasing on* $[0, 1]$ *and* $\mu_3^2(\theta)=u_3^2$ *for* $\theta \geqslant 1$ (*see Fig.* 4.6);

(iii) *if* $n \geqslant 4$, μ_n^k *is strictly increasing on* $[0, \infty)$ *and* $\lim_{\theta \to \infty} \mu_n^k(\theta) = u_n^k$ (*see Fig.* 4.7).

We have also a second function $\bar{\mu}_n^k: (-\infty, 1] \to [1-u_n^k, 1]$ *such that* $\bar{\mu}_n^k(\theta) = 1 - \mu_n^k(1-\theta)$ *for* $\theta \in (-\infty, 1]$.

Proof. For $\theta \in (1, \infty)$ and $\theta \notin \mathbb{Z}$ we proceed as in $n-k$ even case and obtain $0 \leqslant Z_{(0,1)}(N_n^k(\cdot, \theta)) \leqslant 1$. But since $\operatorname{sign}[N_n^k(0, \theta)\, N_n^k(\frac{1}{2}, \theta)] = -1$ then $Z_{(0,1)}(N_n^k(\cdot, \theta)) = 1$. For $\theta \in (0, 1)$ we obtain $S_0^- - S_1^+ = 1$, and consequently $0 \leqslant Z_{(0,1)}(N_n^k(\cdot, \theta)) \leqslant 2$. But $\operatorname{sign}[N_n^k(0, \theta)\, N_n^k(\frac{1}{2}, \theta)] = \operatorname{sign}[N_n^k(\frac{1}{2}, \theta)\, N_n^k(1, \theta)] = -1$ and $Z_{(0,1)}(N_n^k(\cdot, \theta)) = 2$. From Theorem 4.4, we obtain $\mu_2^1(\theta) \in (0, u_2^1)$ for $\theta \in (0, \frac{1}{2})$, $\mu_2^1(\theta) = u_2^1$ for $\theta \in [\frac{1}{2}, 1)$, and for $n \geqslant 3$, $\mu_n^k(\theta) \in (0, u_n^k)$ for $\theta \in (0, 1)$. Lemmas 4.9 and 4.10 complete the proof. ∎

LEMMA 4.9. *The function* μ_n^k *is strictly increasing on the set of non-integer* θ*'s such that*

$$\mu_n^k(\theta) \in \begin{cases} (0, u_n^k) & \text{for} \quad n-k \text{ odd}, \\ (\frac{1}{2}, 1-u_n^k) & \text{for} \quad n-k \text{ even}. \end{cases}$$

Proof. Let us remark that the assumption on the θ's implies that $\mu_n^k(\theta) \in \mathscr{I}_n^k$. We proceed by contradiction to get the result. Let $(\mu_n^k(\theta_1), \theta_1)$

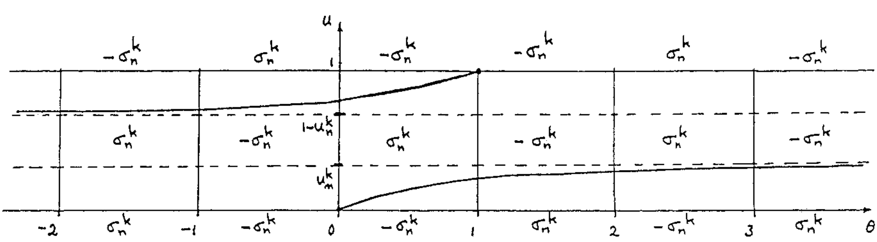

FIG. 4.7. Sketch of μ_n^k for $n \geqslant 4$ and $n-k$ odd.

and $(\mu_n^k(\theta_2), \theta_2)$ be such that $\theta_1 < \theta_2$. The case $\mu_n^k(\theta_1) = \mu_n^k(\theta_2) = u$ is not possible, otherwise θ_1 and θ_2 would be two non-integer zeros of $N_n^k(u, \cdot)$, which is impossible from Theorem 4.5. If $\mu_n^k(\theta_1) > \mu_n^k(\theta_2)$, let $\theta \in (\theta_1, \theta_2)$ such that $\theta \in (l, l+1)$ for some $l \in \mathbb{Z}$. Let $n-k$ even. Since $\psi_n^k(\frac{1}{2}) = \frac{1}{2}$ and $\frac{1}{2} < \theta_1 < \theta_2$, it follows from Theorem 4.6 that

$$\operatorname{sign} N_n^k(\tfrac{1}{2}, \theta) = -\tau(\theta) \operatorname{sign} p_{n-k}(1 - \tfrac{1}{2}, \alpha_n),$$

$$\operatorname{sign} N_n^k(\mu_n^k(\theta_2), \theta) = \tau(\theta) \operatorname{sign} p_{n-k}(\mu_n^k(\theta_2), \alpha_n) = \tau(\theta) \operatorname{sign} p_{n-k}(\tfrac{1}{2}, \alpha_n),$$

$$\begin{aligned}\operatorname{sign} N_n^k(\mu_n^k(\theta_1), \theta) &= -\tau(\theta) \operatorname{sign} p_{n-k}(1 - \mu_n^k(\theta_1), \alpha_n) \\ &= -\tau(\theta) \operatorname{sign} p_{n-k}(\tfrac{1}{2}, \alpha_n).\end{aligned}$$

This implies the existence of at least two zeros for $N_n^k(\cdot, \theta)$ which is impossible. Finally, using $\psi_n^k(0) = 0$, $0 < \theta_1 < \theta_2$, we obtain similarly the result for $n-k$ odd. ▮

LEMMA 4.10. *The functions μ_n^k can be defined for $\theta \in \mathbb{Z}$ in such a way that they are continuous.*

Proof. Fix θ and consider the limits $u_- = \lim_{x \to \theta^-} \mu_n^k(x)$ and $u_+ = \lim_{x \to \theta^+} \mu_n^k(x)$. These limits exist from Lemma 4.9, and $u_- \leqslant u_+$. For $\theta \notin \mathbb{Z}$, from the continuity of $N_n^k(\cdot, \cdot)$, we obtain $N_n^k(u_+, \theta) = 0 = N_n^k(u_-, \theta)$. Then $u_+ = u_- = \mu_n^k(\theta)$ because the zero is unique. For $\theta \in \mathbb{Z}$, if $u_- < u_+$, let $u \in (u_-, u_+)$. Then $\operatorname{sign}[N_n^k(u, \theta - \varepsilon)\, N_n^k(u, \theta + \varepsilon)] = 1$ for $0 < \varepsilon < 1$. Hence θ is a double zero of $N_n^k(u, \cdot)$ and hence $\partial_\theta N_n^k(u, \theta) = 0$. But $\partial_\theta N_n^k(u, \theta)$ is a polynomial of degree $n-k$ in u which is then identically zero on (u_-, u_+), and hence on $(0, 1)$. But this implies θ is at least a double zero for any $u \in (0, 1)$ which is impossible. ▮

As a consequence of the sign structure of $N_n^k(u, \theta)$ and Corollary 2.3(a),(b), the functions μ_n^k are separated as follows (see [9]):

(i) $\mu_n^{n-1}(\theta) < \theta$ for $\theta \in (0, 1]$;

(ii) for $n-k \geqslant 2$, we have $(-1)^{n-k} \mu_n^k(\theta) > (-1)^{n-k} \mu_n^{k+1}(\theta)$ for $\theta \in [0, \infty)$.

Finally from Corollary 2.3(a),(b) and the Implicit Function Theorem, we can show [9] that the function μ_n^k are continuously differentiable functions for $\theta \geqslant 0$ such that $D\mu_n^k(\theta) > 0$ except where μ_n^k is constant. It follows that their inverse functions ψ_n^k are continuously differentiable, strictly increasing on $\mathcal{I}_n^k$, and $D\psi_n^k(u) > 0$.

5. THE CASE $1 \leqslant k \leqslant n-1$: VALUES AND BOUNDS FOR $A_n^k(u, v_n)$ AND $A_n^k(v_n)$

5.1. *Error Bounds.* In this section some results are given in terms of n-degree Euler splines $\mathscr{E}_{n+1}$ (see [13, p. 153; 7]). In particular we will use the inequalities

$$\left|\mathscr{E}_{2n+1}\left(\frac{1}{2}\right)\right| \leqslant 2\left(\frac{2}{\pi}\right)^{2n+1}, \qquad |\mathscr{E}_{2n}(0)| \leqslant 2\,\frac{\pi^2}{8}\left(\frac{2}{\pi}\right)^{2n}, \qquad |\mathscr{E}_{j+1}(v_j)| \geqslant \left(\frac{2}{\pi}\right)^{j} \tag{5.1}$$

and the relation

$$p_n(t, -1) = (-1)^n\, \mathscr{E}_{n+1}(t).$$

THEOREM 5.1. *Let $u \notin \mathscr{I}_n^k$. Then*

$$A_n^k(u, v_n) = |\mathscr{E}_{n-k+2}(u)|/2^{n-k+1}.$$

Since $v_{n-k+1} \notin \mathscr{I}_n^k$, it follows

$$\max_{u \notin \mathscr{I}_n^k} A_n^k(u, v_n) = A_n^k(v_{n-k+1}, v_n).$$

Proof. For $u \notin \mathscr{I}_n^k$ the sign of $N_n^k(u, \theta)$ alternates from one integer to another. Let $f(x) = (1/2^{n+1})\, \mathscr{E}_{n+2}(x)$. Then $f^{(n+1)}(x) = \mathscr{E}_1(x)$ and $\|f^{(n+1)}\|_\infty = 1$. The n-degree interpolating spline of f is $s = 0$. We therefore obtain from (1.2)

$$A_n^k(u, v_n) = \left|\int_{-\infty}^{\infty} N_n^k(u, \theta)\, f^{(n+1)}(\theta)\, d\theta\right| = |e^{(k)}(u)| = |f^{(k)}(u)|$$

$$= \frac{|\mathscr{E}_{n-k+2}(u)|}{2^{n-k+1}}. \quad \blacksquare$$

EXAMPLE 5.2. We have $A_2^1(\frac{1}{2}, \frac{1}{2}) = A_3^2(\frac{1}{2}, 0) = \cdots = A_n^{n-1}(\frac{1}{2}, v_n) = \frac{1}{8}$, and $A_3^1(0, 0) = A_4^2(0, \frac{1}{2}) = \cdots = A_n^{n-2}(0, v_n) = \frac{1}{24}$, since $A_n^{n-1}(\frac{1}{2}, v_n) = \mathscr{E}_3(\frac{1}{2})/2^2$, $A_n^{n-2}(0, v_n) = \mathscr{E}_4(0)/2^3$, and $\mathscr{E}_1(u) = 1$, $\mathscr{E}_2(u) = 2u - 1$, $\mathscr{E}_3(u) = 2(u^2 - u)$, $\mathscr{E}_4(u) = (4u^3/3) - 2u^2 + \frac{1}{3}$ for $u \in [0, 1]$.

LEMMA 5.3. *Let* $\theta \notin \mathbb{Z}$.

(i) *For* $n-k$ *odd and* $\theta > 0$, $|N_n^k(0, \theta)| > |N_n^k(u, \theta)|$ *for any* $u \in (0, \mu_n^k(\theta)]$.

(ii) *For* $n-k$ *even and* $\theta > \frac{1}{2}$, $|N_n^k(\frac{1}{2}, \theta)| > |N_n^k(u, \theta)|$ *for any* $u \in (\frac{1}{2}, \mu_n^k(\theta)]$.

Proof. (i) Since sign $N_n^{k+1}(u, \theta)$ is constant on $(0, \mu_n^{k+1}(\theta))$ and $\mu_n^k(\theta) \leqslant \mu_n^{k+1}(\theta)$, it follows from Corollary 2.3(a),(b) that sign $\partial_u N_u^k(u, \theta)$ is constant on $(0, \mu_n^k(\theta))$. Then the maximum of $|N_n^k(u, \theta)|$ on $[0, \mu_n^k(\theta)]$ is at $u = 0$ because $N_n^k(\mu_n^k(\theta), \theta) = 0$. (ii) The proof is similar. ∎

THEOREM 5.4. *For* $u \in \mathscr{I}_n^k$, *we have* $A_n^k(u, v_n) \leqslant A_n^k(v_{n-k+1}, v_n) + A_n^k(v_{n-k}, v_n)$.

Proof. Let $u \in \mathscr{I}_n^k$. Then

$$A_n^k(u, v_n) = \left| \int_{-\infty}^{\infty} N_n^k(u, \theta)\, \mathscr{E}_1(\theta)\, d\theta \right| + 2 \int_{\psi_n^k(u)}^{\infty} |N_n^k(u, \theta)|\, d\theta$$

$$= \frac{|\mathscr{E}_{n-k+2}(u)|}{2^{n-k+1}} + 2 \int_{\psi_n^k(u)}^{\infty} |N_n^k(u, \theta)|\, d\theta.$$

For $n-k$ odd, we consider $u \in [0, u_n^k)$. Since $|N_n^k(0, \theta)| \geqslant |N_n^k(u, \theta)|$ for $\theta \geqslant \psi_n^k(u)$ from Lemma 5.3, we have

$$A_n^k(u, v_n) \leqslant \frac{|\mathscr{E}_{n-k+2}(u_n^k)|}{2^{n-k+1}} + 2 \int_0^{\infty} |N_n^k(0, \theta)|\, d\theta \leqslant \frac{|\mathscr{E}_{n-k+2}(\frac{1}{2})|}{2^{n-k+1}} + A_n^k(0, v_n).$$

For $n-k$ even, we consider $u \in [\frac{1}{2}, 1 - u_n^k)$. Since $|N_n^k(\frac{1}{2}, \theta)| \geqslant |N_n^k(u, \theta)|$ for $\theta \geqslant \psi_n^k(u)$ from Lemma 5.3, we have

$$A_n^k(u, v_n) \leqslant \frac{|\mathscr{E}_{n-k+2}(1 - u_n^k)|}{2^{n-k+1}} + 2 \int_{1/2}^{\infty} |N_n^k(\tfrac{1}{2}, \theta)|\, d\theta$$

$$\leqslant \frac{|\mathscr{E}_{n-k+2}(1)|}{2^{n-k+1}} + A_n^k(\tfrac{1}{2}, v_n). \quad ∎$$

We obtain (1.12) from Theorems 5.1 and 5.4.

Remark 5.5 (Proof of (1.13) and (1.14)). (a) Let $n-k$ odd. Then $v_{n-k} = 0$, $v_{n-k+1} = \frac{1}{2}$, and from Theorem 5.1

$$A_n^k(\tfrac{1}{2}, v_n) = |\mathscr{E}_{n-k+2}(\tfrac{1}{2})| / 2^{n-k+1}.$$

Also

$$A_n^k(0, v_n) = 2 \int_0^\infty |N_n^k(0, \theta)| \, d\theta.$$

From the sign properties of $N_n^k(0, \cdot)$ (see Fig. 4.1), and G_j given by (2.7)–(2.8), we obtain (1.13).

(b) Let $n-k$ even. Then $v_{n-k} = \frac{1}{2}$, $v_{n-k+1} = 0$, and from Theorem 5.1

$$A_n^k(0, v_n) = |\mathscr{E}_{n-k+2}(0)|/2^{n-k+1}.$$

Also

$$A_n^k(\tfrac{1}{2}, v_n) = 2 \int_{1/2}^\infty |N_n^k(\tfrac{1}{2}, \theta)| \, d\theta = 2 \int_{1/2}^1 |N_n^k(\tfrac{1}{2}, \theta)| \, d\theta + 2 \int_1^\infty |N_n^k(\tfrac{1}{2}, \theta)| \, d\theta.$$

From the sign properties of $N_n^k(\frac{1}{2}, \cdot)$ (see Fig. 4.2), we have (1.14).

Let us observe that, from (5.1),

$$A_n^k(v_n) \geqslant A_n^k(v_{n-k+1}, v_n) = \frac{|\mathscr{E}_{n-k+2}(v_{n-k+1})|}{2^{n-k+1}} \geqslant \left(\frac{2}{\pi}\right)^{n-k+1} \frac{1}{2^{n-k+1}} = \frac{1}{\pi^{n-k+1}}$$

which is the lower estimate for (1.15). The upper estimate in (1.15) is established in Theorem 5.7.

THEOREM 5.6. *For any* $0 \leqslant k \leqslant n-1$ *we have*

$$A_n^k(v_n) \leqslant A_n^k(v_{n-k+1}, v_n) + \tfrac{1}{2} A_n^{k+1}(v_n). \tag{5.2}$$

Proof. Using the regularity of N_n^k and ψ_n^k, and Corollary 2.3(a),(b), we obtain from (1.6)

$$D_u A_n^k(u, v_n) = \int_{-\infty}^\infty D_u \, |N_n^k(u, \theta)| \, d\theta.$$

Hence

$$|D_u A_n^k(u, v_n)| \leqslant \int_{-\infty}^\infty |N_n^{k+1}(u, \theta)| \, d\theta = A_n^{k+1}(u, v_n) \leqslant A_n^{k+1}(v_n). \tag{5.3}$$

Then, for any $u \in [0, \frac{1}{2}]$, (5.2) follows. ∎

THEOREM 5.7. *For* $1 \leqslant k \leqslant n-1$, *we have*

(i) *for* $n-k$ *odd*, $$A_n^k(v_n) < \left[\frac{1}{2} + \frac{\sqrt{n+1}}{\pi^{1/2}}\right] \frac{1}{\pi^{n-k}} \leqslant \frac{\sqrt{n+1}}{\pi^{n-k}};$$

(ii) *for* $n-k$ *even*, $$A_n^k(v_n) \leqslant \left[\frac{2+\pi}{4} + \frac{\pi^{1/2}}{2}\sqrt{n+1}\right] \frac{1}{\pi^{n-k}} \leqslant 2\,\frac{\sqrt{n+1}}{\pi^{n-k}}.$$

Proof. (i) Using (1.13), we obtain

$$A_n^k(0, v_n) \leqslant \frac{2}{n+1}\left(\sum_{\alpha \in \mathscr{A}_n(v_n)} \frac{1-\alpha}{1+\alpha}\right) \max_{\alpha \in \mathscr{A}_n(v_n)} \left|\frac{p_{n-k}(1, \alpha)}{(1-\alpha)^{n-k+1}}\right|, \tag{5.4}$$

and from (3.5) and (3.6) we have

$$\frac{2}{n+1} \sum_{\alpha \in \mathscr{A}_n(v_n)} \frac{1-\alpha}{1+\alpha} \leqslant \frac{4}{\pi^{3/2}}\sqrt{n+1}.$$

But if we consider

$$\Phi_m(1-t, x) = e^{(1-t)x}\,\frac{p_m(t, -e^x)}{(1+e^x)^{m+1}},$$

it can be shown [9] that for m odd

$$|\Phi_m(0, x)| \leqslant |\Phi_m(0, 0)| = \frac{|p_m(1, -1)|}{2^{m+1}} = \frac{|\mathscr{E}_{m+1}(0)|}{2^{m+1}}.$$

Then

$$\max_{\alpha \in \mathscr{A}_n(v_n)} \left|\frac{p_{n-k}(1, \alpha)}{(1-\alpha)^{n-k+1}}\right| \leqslant \frac{|\mathscr{E}_{n-k+1}(0)|}{2^{n-k+1}}.$$

Hence, from (1.12) and Remark 5.5

$$A_n^k(v_n) \leqslant \frac{|\mathscr{E}_{n-k+2}(1/2)|}{2^{n-k+1}} + \frac{\sqrt{n+1}}{\pi^{3/2}}\,\frac{|\mathscr{E}_{n-k+1}(0)|}{2^{n-k-1}}.$$

Using $\mathscr{E}_{j+1}(v_j) \leqslant (2/\pi)^{j-1}$, the result follows. (ii) Using (1.14), Theorem 5.6, and part (i) we obtain (ii). ∎

5.2. *A Special Result for the Case* $k = n-1$. Using the derivative of ψ_n^{n-1}, we have the following slight improvement.

THEOREM 5.8. $\max\{A_n^{n-1}(0, v_n), 1/8\} \leqslant A_n^{n-1}(v_n) \leqslant \max\{A_n^{n-1}(0, v_n), (u_n^{n-1}/2), 1/8\}$.

Proof. From Theorem 5.4, for $u \in \mathcal{I}_n^{n-1}$, $A_n^{n-1}(u, v_n) = |\mathcal{E}_3(u)|/8 + 2W_n^{n-1}(u, \psi_n^{n-1}(u))$ where

$$W_n^l(u, y) = \int_y^\infty |N_n^l(u, \theta)| \, d\theta.$$

It follows $D_u W_n^{n-1}(u, \psi_n^{n-1}(u)) = -W_n^n(u, \psi_n^{n-1}(u)) \leqslant 0$ and $D_u^2 W_n^{n-1}(u, \psi_n^{n-1}(u)) = |N_n^n(u, \psi_n^{n-1}(u))| \, D_u \psi_n^{n-1}(u) \geqslant 0$. Then $W_n^{n-1}(u, \psi_n^{n-1}(u))$ is a decreasing continuous convex function on $[0, u_n^{n-1})$ and $\lim_{u \to u_n^{n-1}} W_n^{n-1}(u, \psi_n^{n-1}(u)) = 0$. Hence for $u \in [0, u_n^{n-1})$

$$A_n^{n-1}(u, v_n) \leqslant \frac{u}{2} + 2W_n^{n-1}(u, \psi_n^{n-1}(u))$$

$$\leqslant \max\left\{2W_n^{n-1}(0, \psi_n^{n-1}(0)), \frac{u_n^{n-1}}{2}\right\} = \max\left\{A_n^{n-1}(0, v_n), \frac{u_n^{n-1}}{2}\right\}.$$

The result follows since $A_n^{n-1}(u, v_n) \leqslant \frac{1}{8} = A_n^{n-1}(\frac{1}{2}, v_n)$ for $u \in [u_n^k, \frac{1}{2}]$. ∎

EXAMPLE 5.9. Values of $A_n^{n-1}(v_n)$ *for* $n = 2$ *and* 3.

(a) $n = 2$: $\max\{A_2^1(0, \frac{1}{2}), \frac{1}{8}\} \leqslant A_2^1(\frac{1}{2}) \leqslant \max\{A_2^1(0, \frac{1}{2}), u_2^1/2, \frac{1}{8}\}$. Since $A_2^1(0, \frac{1}{2}) = 2\,|\alpha_2|/3(1+\alpha_2)(1-\alpha_2)$ where $\alpha_2 = -3 + 2\sqrt{2}$, and $u_2^1 = (2-\sqrt{2})/4$, we have $A_2^1(0, \frac{1}{2}) = 1/6\sqrt{2}$, $u_2^1/2 = (2-\sqrt{2})/8$, and then $A_2^1(\frac{1}{2}) = \frac{1}{8}$.

(b) $n = 3$: $\max\{A_3^2(0, 0), \frac{1}{8}\} \leqslant A_3^2(0) \leqslant \max\{A_3^2(0, 0), u_3^2/2, \frac{1}{8}\}$. Since $A_3^2(0, 0) = 2\,|\alpha_3|/4(1+\alpha_3)(1-\alpha_3)$ where $\alpha_3 = -2 + \sqrt{3}$, and $u_3^2 = (3-\sqrt{3})/6$, we have $A_3^2(0, 0) = \sqrt{3}/12$, $u_3^2/2 = (3-\sqrt{3})/12$, and then $A_3^2(0) = \sqrt{3}/12$.

5.3. *A Result for Low Values of k.* The bounds given by Theorem 5.7 are not the best possible for low values of k, in fact those in (1.10) are better. The next result improves these bounds under a condition on u_n^k and u_n^{k+1}.

THEOREM 5.10. *Let* $1 \leqslant k \leqslant n-1$. *If* $(-1)^{n-k+1}(u_n^{k+1} - u_n^k) > 0$, *then*

$$A_n^k(v_n) \leqslant \max\left\{2\frac{|\mathcal{E}_{n-k+2}(u_n^k)|}{2^{n-k+1}}, \frac{|\mathcal{E}_{n-k+2}(v_n^{k+1})|}{2^{n-k+1}}\right\}$$

and $A_n^k(v_n) \leqslant 1/\pi^{n-k}$.

Proof. Let n be odd. We know that $A_n^k(u, v_n) = |\mathcal{E}_{n-k+2}(u)|/2^{n-k+1}$ for $u \in [u_n^k, \frac{1}{2}]$. Also, from (5.3), $|D_u A_n^k(u, v_n)| \leqslant A_n^{k+1}(v_n)$ for $u \in [0, u_n^k)$. But for $u \in [0, u_n^{k+1})$, $A_n^{k+1}(u, v_n) = |\mathscr{E}_{n-k+1}(u)|/2^{n-k}$. Hence, if $u_n^k < u_n^{k+1}$, for $u \in [0, u_n^k)$ we have

$$A_n^k(u, v_n) \leqslant A_n^k(u_n^k, v_n) + \int_0^{u_n^k} A_n^{k+1}(u, v_n)\, du = 2\, \frac{|\mathscr{E}_{n-k+2}(u_n^k)|}{2^{n-k+1}}$$

and the result follows. The even case can be proved similarly. Finally, the last bound follows from (5.1). ∎

EXAMPLE 5.11. For $n=3$ and $k=1$, we have $A_3^1(0, 0) = \frac{1}{24}$ from Example 5.2. Since $u_3^1 = \frac{1}{2} - (\sqrt{2} - 1)\sqrt{3}/6$ and $u_3^2 = \frac{1}{2} - (\sqrt{3}/6)$, we have $u_3^2 < u_3^1$ and we can apply Theorem 5.10. Since $\mathscr{E}_4(u) = (4u^3 - 6u^2 + 1)/3$, we obtain the desired result.

Examples 3.2, 5.9, and 5.11 complete the proofs of the values given in Example 1.1.

6. CONCLUSION

In this paper we have analyzed the Peano kernel $N_n^k(u, v_n, \theta)$ related to the cardinal spline interpolation problem. We have obtained a new representation for this kernel and investigated the nature of its zeros. We have applied these results to get exact explicit expressions for $A_n^k(u, v_n)$ and $A_n^k(v_n)$, and bounds for $A_n^k(u, v_n)$ and $A_n^k(v_n)$ (optimal only for $k = n$). We have obtained good bounds for $A_n^k(v_n)$ for $k = 1, ..., n-1$, but exact explicit expressions remain to be found in that case. To be more specific, we could improve the upper bounds in (1.15) if a sharper inequality is obtained for (5.4).

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